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ALGEBRAS IN SETS OF QUEER FUNCTIONS

BY

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ABSTRACT

We show that the set of continuous nowhere differentiable functions, the set of Dirichlet series which are bounded in the right half-plane and diverge everywhere on the imaginary axis, and the set of continuous interpolating functions contain big algebras.

1. Introduction

Many examples of functions verifying some pathological properties have appeared in analysis: continuous nowhere differentiable functions, universal functions, etc. In many situations, pathology is the rule: an application of Baire's Theorem shows that in a suitable topological space, all elements of a dense G_δ-set share this pathological behavior.

In the last few years, it has been observed that these sets of pathological functions are large in an algebraic sense: they contain a linear subspace (of large dimension), which is sometimes dense or closed, except, of course, for the

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zero function: see for instance [2], [5], [6], [9], [10], [12], [13], [15], [19], and [1] for a first survey.

The next step is to study if these sets can contain an algebra. This has been done in [4] for the set of everywhere surjective functions on $\mathbb{C}$ and in [3] for the set of non-convergent Fourier series. In the first case, an algebra is constructed using rings of polynomials. In the second, it is produced by a clever construction using series. This type of construction has already been used to obtain subspaces (see [5] for instance). Of course, getting an algebra is more difficult, and the proof of [3] is very technical.

In this paper, we give three more examples for the existence of an algebra. We begin with the set of continuous nowhere differentiable functions. Our proof is explicit and uses nowhere Hölder functions with control. Next, we study the set of Dirichlet series which are bounded in the right half-plane and diverge everywhere on the imaginary axis. This example is in some sense relatively close to that given in [3]. However, our method is completely different, and it is very surprising: we use topological arguments (namely Baire's Theorem) to produce an algebra! Our last example concerns continuous interpolating functions. It answers a question suggested to the second author by Y. Benyamini.

2. Continuous nowhere differentiable functions

The algebraic structure of the set of continuous nowhere differentiable functions has attracted the attention of many mathematicians (see [9], [12], [15], [18] or [19], for instance). In particular, it is now well-known that the set of continuous nowhere differentiable functions on $[0,1]$ contains a closed subspace and a dense subspace of $\mathcal{C}([0,1])$, which are both infinitely generated. We solve here the question of the existence of an algebra. In fact, our result is even better since the forthcoming theorem addresses the question of nowhere Hölder functions.

Theorem 1: *The set of continuous functions on* $[0,1]$ *which are nowhere Hölder contains a non-trivial algebra (except for the zero-function).*

Proof: The proof follows almost immediately from the existence of a single continuous nowhere Hölder function on $[0,1]$. Indeed, let u be such a function. We claim that any non-zero function f which belongs to the algebra $A(u)$ generated by u is nowhere differentiable. We write $f=\lambda_1 u+\cdots+\lambda_m u^m$ with $\lambda_m\neq 0$. Fix $x\in[0,1]$, and for $y\in[0,1], y\neq x$, denote by $\Delta(y)$ the difference $f(y)-f(x)$. Using Taylor's formula for polynomials, we get:

$$\Delta(y)=(u(y)-u(x))^k(\lambda+P(x,y)),$$

where $\lambda \neq 0$ and $P(x,x)=0$. It suffices to use that u is nowhere α/k-Hölder (with $\alpha>0$) to conclude that f cannot be $\alpha-$Hölder at x. ∎

Remark 2: For several topological notions of the size of sets (categories, porosity, Haar null sets), the set of nowhere Hölder functions is big in $\mathcal{C}([0,1])$ (see [18]). Thus, for "many" continuous functions u, the algebra generated by u consists of nowhere Hölder functions.

Observe that the algebra exhibited in the previous proof is generated (as an algebra) by a single function. We are able to improve this statement.

THEOREM 3: *The set of continuous nowhere Hölder functions on $[0,1]$ contains a dense algebra, with infinitely many algebraically independent generators.*

Proof: First of all, we need to construct a sequence of nowhere Hölder functions, with control. Let us introduce some notation.

- For two sequences of real numbers (a_n) and (b_n), $a_n \gg b_n$ means that a_n/b_n goes to infinity as n goes to infinity.
- If $\beta=(\beta_1,\dots,\beta_r)$ and $\gamma=(\gamma_1,\dots,\gamma_r)$ are two r-tuples in $\mathbb{N}_0^r=(\mathbb{N}\backslash\{0\})^r$, we say that $\beta>\gamma$ if there exists $1\leq m\leq r$ with $\beta_r=\gamma_r,\dots,\beta_{m+1}=\gamma_{m+1}$ and $\beta_m<\gamma_m$ (namely, we order r-tuples of indices in the reverse lexicographical order).

LEMMA 4: *There exists a sequence $(u_k)_{k\geq 1}$ of elements of $\mathcal{C}([0,1])$ satisfying the following property: given $r\geq 1$ distinct integers $1\leq i_1<\cdots<i_r$, given $x\in[0,1]$, there exists a sequence of real numbers (h_n) converging to 0, where*

(a) *For any $\beta\in\mathbb{N}_0^r$ and any $\alpha>0$,*

$$\frac{|u_{i_1}(x+h_n)-u_{i_1}(x)|^{\beta_1}\cdots|u_{i_r}(x+h_n)-u_{i_r}(x)|^{\beta_r}}{h_n^\alpha}\xrightarrow{n\to\infty}\infty.$$

(b) *For any $\beta\in\mathbb{N}_0^r$ and any $j\notin\{i_1,\dots,i_r\}$,*

$$|u_{i_1}(x+h_n)-u_{i_1}(x)|^{\beta_1}\cdots|u_{i_r}(x+h_n)-u_{i_r}(x)|^{\beta_r}\gg|u_j(x+h_n)-u_j(x)|.$$

(c) *For any $\beta,\gamma\in\mathbb{N}_0^r$ with $\beta>\gamma$,*

$$|u_{i_1}(x+h_n)-u_{i_1}(x)|^{\beta_1}\cdots|u_{i_r}(x+h_n)-u_{i_r}(x)|^{\beta_r}$$
$$\gg|u_{i_1}(x+h_n)-u_{i_1}(x)|^{\gamma_1}\cdots|u_{i_r}(x+h_n)-u_{i_r}(x)|^{\gamma_r}.$$

Proof of Lemma 4: Let u be the Van der Waerden's function, $u(x)=x$ for $x\in[0,1/2]$, $u(x)=1-x$ for $x\in[1/2,1]$ and $u(x+1)=u(x)$. We define u_k by setting

$$u_k(x) := \sum_{m=0}^{+\infty} \frac{1}{10^{m^k}} u(10^{(p_k m)!}x),$$

where p_k is the k-th prime number. Pick now any $x \in [0,1]$, let $1 \leq i_1 < i_2 < \cdots < i_r$, and let $n \geq 1$. To simplify notation, p'_j will stand for $\frac{p_{i_1}\cdots p_{i_r}}{p_j}$, whereas $u_{k,m}$ will denote the function $y \mapsto u(10^{(p_k m)!}y)$. Clearly, there exists $\varepsilon_n = \pm 1$ such that, setting $h_n := \varepsilon_n / 10^{(p_{i_1}\cdots p_{i_r} p_n)!+1}$, then $x + h_n \in [0,1]$ and

$$|u(10^{(p_{i_1}\cdots p_{i_r} p_n)!}(x+h_n)) - u(10^{(p_{i_1}\cdots p_{i_r} p_n)!}x)| = 1/10.$$

Let us estimate $u_{i_k}(x+h_n) - u_{i_k}(x)$. Observe that $u_{k,m}$ is $1/10^{(p_k m)!}$-periodic, and $10^{(p_k m)!}$-Lipschitz. Hence, as soon as $m > p'_{i_k} p_n$, the difference $u_{i_k,m}(x+h_n) - u_{i_k,m}(x)$ is equal to zero. In particular, one has

$$\begin{aligned}
|u_{i_k}(x+h_n) - u_{i_k}(x)| \geq & \frac{1}{10^{(p'_{i_k} p_n)^{i_k}}} |u_{i_k, p'_{i_k} p_n}(x+h_n) - u_{i_k, p'_{i_k} p_n}(x)| \\
& - \sum_{m=0}^{p'_{i_k} p_n - 1} |u_{i_k,m}(x+h_n) - u_{i_k,m}(x)| \\
\geq & \frac{1}{10^{(p'_{i_k} p_n)^{i_k}+1}} - p'_{i_k} p_n \times \frac{10^{(p_{i_1}\cdots p_{i_r} p_n - 1)!}}{10^{(p_{i_1}\cdots p_{i_r} p_n)!}}.
\end{aligned}$$

The same computation gives also an upper estimate, and we finally obtain the existence of a constant C_k such that

$$u_{i_k}(x+h_n) - u_{i_k}(x) \sim \frac{C_k}{10^{(p'_{i_k} p_n)^{i_k}}}.$$

In particular, given $\beta \in \mathbb{N}_0^r$, one has

$$(u_{i_1}(x+h_n) - u_{i_1}(x))^{\beta_1} \cdots (u_{i_r}(x+h_n) - u_{i_r}(x))^{\beta_r} \sim \frac{C_\beta}{10^{\beta_1 (p'_{i_1} p_n)^{i_1} + \cdots + \beta_r (p'_{i_r} p_n)^{i_r}}}.$$

This gives (a) and (c) of the lemma. It remains to prove (b), and, in particular, to estimate $u_j(x+h_n) - u_j(x)$, provided $j \neq i_k$. As previously, using periodicity, the sum is finite:

$$|u_j(x+h_n) - u_j(x)| \leq \sum_{m=0}^{[p'_j p_n]} |u_{j,m}(x+h_n) - u_{j,m}(x)|.$$

The key argument here is the fact that $p'_j p_n$ is not an integer, which, in particular, implies $p_j [p'_j p_n] \leq p_{i_1} \cdots p_{i_r} p_n - 1$. This gives the inequality

$$|u_j(x+h_n) - u_j(x)| \leq p_{i_1} \cdots p_{i_r} p_n \times \frac{10^{(p_{i_1}\cdots p_{i_r} p_n - 1)!}}{10^{(p_{i_1}\cdots p_{i_r} p_n)!}} \leq \frac{p_{i_1} \cdots p_{i_r} p_n}{10^{\frac{1}{2}(p_{i_1}\cdots p_{i_r} p_n)!}},$$

which in turn shows immediately that (b) holds. ∎

Observe now that if (u_k) is a sequence of functions satisfying the conclusions of Lemma 4, if (g_k) is any sequence of polynomials, and if (δ_k) is any sequence of positive numbers, then the sequence $(g_k+\delta_k u_k)$ keeps on satisfying the same conclusions. Hence, we may assume that (u_k) is dense in $\mathcal{C}([0,1])$, and consider A the (dense) algebra generated by the u_k's. Pick $f\in A\backslash\{0\}$, f may be written as $f=Q(u_1,\dots,u_m)$ where Q is a non-zero polynomial. Using Taylor's formula, for a given $x\in[0,1]$ and for any $y\in[0,1]$, we may write

$$\Delta(y)=f(y)-f(x)=\sum_{\lambda\in\Lambda}a_\lambda(u_1(y)-u_1(x))^{\lambda_1}\cdots(u_d(y)-u_d(x))^{\lambda_d},$$

where $\Lambda\subset\mathbb{N}^d$ is finite and $a_\lambda\neq 0$. Let us set

$$r:=\min\{\mathrm{card}\{i\colon \lambda_i\neq 0\}\colon \lambda\in\Lambda\}>0,$$

and let $\Lambda':=\{\lambda\in\Lambda\colon \mathrm{card}\{i\colon \lambda_i\neq 0\}=r\}$. In Λ', we choose the element λ^0 which is maximal for the reverse lexicographical order of $d-$tuples. Thus, we may decompose $\Delta(y)$ into

$$\begin{aligned}\Delta(y)=&a_{\lambda^0}(u_1(y)-u_1(x))^{\lambda_1^0}\cdots(u_d(y)-u_d(x))^{\lambda_d^0}\\&+\sum_{\lambda\in\Lambda'\backslash\{\lambda^0\}}a_\lambda(u_1(y)-u_1(x))^{\lambda_1}\cdots(u_d(y)-u_d(x))^{\lambda_d}\\&+\sum_{\lambda\in\Lambda\backslash\Lambda'}a_\lambda(u_1(y)-u_1(x))^{\lambda_1}\cdots(u_d(y)-u_d(x))^{\lambda_d}\\:=&\Delta_1(y)+\Delta_2(y)+\Delta_3(y).\end{aligned}$$

Rewrite now $\Delta_1(y)$ as $a_{\lambda^0}(u_{i_1}(y)-u_{i_1}(x))^{\beta_1}\cdots(u_{i_r}(y)-u_{i_r}(x))^{\beta_r}$, where the β_i's are non-zero, and the sequence (i_k) is increasing. Lemma 4 gives a sequence (h_n) associated to $i_1,\dots,i_r$. Then, by (b), $\Delta_1(x+h_n)\gg\Delta_3(x+h_n)$, and by (b) and (c), $\Delta_1(x+h_n)\gg\Delta_2(x+h_n)$. On the other hand, (a) shows that for any $\alpha>0$, $\lim_{+\infty}|\Delta_1(x+h_n)/h_n^\alpha|=+\infty$. Thus, f is not $\alpha-$Hölder at x.

Finally, it remains to prove that we cannot extract from the sequence (u_k) a finite system of generators $(u_{j_1},\dots,u_{j_r})$. Suppose that the contrary holds, let $i\notin\{j_1,\dots,j_r\}$, and $x\in[0,1]$. Using the sequence (h_n) given by Lemma 4 for this single i, and using (b), it is easy to check that u_i cannot belong to the algebra generated by $u_{j_1},\dots,u_{j_r}$. ∎

3. Divergent Dirichlet series

Let $\mathcal{H}^\infty$ be the set of analytic functions defined on the half-plane $\mathbb{C}_+ := \{s \in \mathbb{C}\colon \Re(s) > 0\}$, which are bounded on $\mathbb{C}_+$ and which can be represented by a Dirichlet series $f(s) = \sum_{k\geq 1} a_k k^{-s}$ convergent in some half-plane. $\mathcal{H}^\infty$ is a Banach space if it is endowed with the sup-norm, and classical results about the abscissa of convergence of products of Dirichlet series ensure that $\mathcal{H}^\infty$ is an algebra.

If $f \in \mathcal{H}^\infty$ can be represented by a Dirichlet series $f(s) = \sum_{k\geq 1} a_k k^{-s}$ convergent in some half-plane, a theorem of Bohr ensures that this series converges in fact in $\mathbb{C}_+$. Thus a natural question, first asked by Hedenmalm in [14], arises: what can be said about the convergence on the boundary $i\mathbb{R}$? The answer, which might be surprising if we compare it with the classical Theorem of Carleson on Fourier series, was given in [7]: there exists a Dirichlet series $f(s) = \sum_{k\geq 1} a_k k^{-s} \in \mathcal{H}^\infty$ such that $\sum a_k k^{it}$ diverges for any $t \in \mathbb{R}$.

The algebraic structure of the set of such series has been investigated in [5] and in [6]. In particular, it contains a closed infinite-dimensional subspace of $\mathcal{H}^\infty$. Our aim here is to prove the following theorem.

THEOREM 5: *The set of Dirichlet series, which are bounded in the right half-plane and diverge everywhere on the imaginary axis, contains a non-trivial algebra (except for the zero-function).*

Before we proceed with the proof of this theorem, let us review some results and notation on Dirichlet series. For a Dirichlet series $f = \sum_{k=1}^{+\infty} a_k k^{-s}$, we will denote by $S_n(f,s)$ its partial sum $\sum_{k=1}^{n} a_k k^{-s}$. For notational convenience, if $s = it$ belongs to $i\mathbb{R}$, we will write $S_n(f,t)$ instead of $S_n(f,it)$. A Dirichlet polynomial $P = \sum_{k=1}^{K} a_k k^{-s}$ is a Dirichlet series whose sum is finite. Its degree $\deg(P)$ is given by $\sup\{k; a_k \neq 0\}$, and its valuation $\operatorname{val}(P)$ is defined by $\inf\{k; a_k \neq 0\}$. Obviously, $\deg(PQ) = \deg(P)\deg(Q)$, and the same formula holds true for the valuation.

To construct a subspace or an algebra of (everywhere) divergent Dirichlet (Fourier) series, we need a basic lemma, which is generally obtained for the construction of a single example. In our context, the following result which comes from [7] will be sufficient:

LEMMA 6: *Given any $a > 0$ and any $M > 0$, there exists a Dirichlet polynomial $Q = \sum_{k=1}^{+\infty} a_k k^{-s}$ and an integer K such that $\| Q\|_\infty \leq 1$ and*

$$\left|\sum_{k=1}^{K} a_k k^{it}\right| \geq M, \quad \text{for any } t \in [-a,a].$$

If we replace Q by $2^{-s}Q$, the conclusion remains true. Hence, we may assume that $\mathrm{val}(Q) \geq 2$. Finally, let $\mathcal{C}$ be the closure in $\mathcal{H}^\infty$ of the set of Dirichlet polynomials (recall that $\mathcal{H}^\infty$ is not separable). Obviously, $\mathcal{C}$ is a Baire space. We are now ready for the proof.

Proof of Theorem 5: For $m \geq 1$, $\alpha \in \mathbb{C}^m$ and $f \in \mathcal{C}$, we set $f_\alpha := \alpha_1 f + \cdots + \alpha_m f^m$. We introduce, for $u \geq 1$, $s \geq 1$ and $m \geq 1$, the sets

$$\begin{aligned}\mathcal{D}(u,s,m) := \{f \in \mathcal{C} \colon \forall \alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{C}^m \text{ with}\\ \sup |\alpha_i| \leq s \text{ and } |\alpha_m| \geq 1/s,\\ \text{there exist } p, q \in \mathbb{N} \text{ such that}\\ |S_p(f_\alpha, t) - S_q(f_\alpha, t)| > u \text{ for any } t \in [-u, u]\}\end{aligned}$$

We will be concerned with

$$\mathcal{D} := \bigcap_{u \geq 1} \bigcap_{s \geq 1} \bigcap_{m \geq 1} \mathcal{D}(u,s,m).$$

We claim that if f belongs to $\mathcal{D}$, the algebra $A(f)$ generated by f solves the problem. Indeed, pick any $g \in A(f) \backslash \{0\}$, which may be written as f_α for certain $m \geq 1$ and $\alpha \in \mathbb{C}^m$, with $\alpha_m \neq 0$. We choose s with $|\alpha_m| \geq 1/s$ and $\sup |\alpha_i| \leq s$. Since f belongs to $\mathcal{D}$ and, in particular, to $\bigcap_{u \geq 1} \mathcal{D}(u,s,m)$, for any $t \in \mathbb{R}$ and any $u \geq 1$ there exist p, q with $|S_p(g,t) - S_q(g,t)| \geq u$. This proves the divergence of $\big(S_n(g,t)\big)_{n \geq 1}$.

Hence, we are reduced to prove that $\mathcal{D}$ is not empty. This will be done via Baire's Category Theorem and the two following facts:

Fact 1: For any u, s, m, $\mathcal{D}(u,s,m)$ is open.

Taking the complement, this follows easily from the compactness of

$$\{(\alpha_1, \dots, \alpha_m) \in \mathbb{C}^m \colon \sup |\alpha_i| \leq s \text{ and } |\alpha_m| \geq 1/s\}$$

and from the continuity of $S_p(f_\alpha, .)$ in f and α.

Fact 2: For any u, s, m, $\mathcal{D}(u,s,m)$ is dense.

Let us fix a Dirichlet polynomial P and $\varepsilon > 0$. We write the Taylor series of $(1+x)^{1/m}$ as $1 + \sum_{l=1}^{+\infty} \beta_l x^l$. Let $C := 1 + \sum_{l=1}^{+\infty} |\beta_l| (\frac{1}{2})^l$. Lemma 6 gives a Dirichlet polynomial Q and an integer K with $\mathrm{val}(Q) \geq 2$, $\|Q\|_\infty \leq 1/2$ and

$$\Big|\sum_{k=2}^{K} a_k k^{it}\Big| > \Big(\frac{C}{\varepsilon}\Big)^m \times s \times u \quad \text{for any } t \in [-u, u].$$

Observe that, for any $L\geq 1$, there exist complex numbers $(\gamma_l)_{l\geq L+1}$ such that:

$$\left(1+\sum_{l=1}^{L}\beta_l x^l\right)^m = 1+x+\sum_{l=L+1}^{mL}\gamma_l x^l.$$

In particular, since $\mathrm{val}(Q)\geq 2$, we may choose $L\geq 1$ such that

$$\left(1+\sum_{l=1}^{L}\beta_l Q^l\right)^m = 1+Q+R,$$

with $\mathrm{val}(R)>\deg(Q)$. We then fix $n>\max(\deg(P),\deg(R))$ and we set

$$f=P+\frac{\varepsilon}{C}n^{-s}\left(1+\sum_{l=1}^{L}\beta_l Q^l\right)=P+n^{-s}T.$$

Observe that $\|f-P\|_\infty\leq\varepsilon$. Therefore, f is close enough to P, and it remains to prove that it belongs to $\mathcal{D}(u,s,m)$. Let us define $p=2n^m$ and $q=n^m\times K$. For $j<m$, a straightforward computation shows that $\deg(f^j)\leq n^j\times\deg(T^j)<p$. In particular, $S_p(f^j,t)-S_q(f^j,t)=0$. On the other hand, one has

$$f^m=\underbrace{\sum_{k=0}^{m-1}C_m^k P^{m-k}(n^{-s})^k T^k}_{\deg<p}+\left(\frac{\varepsilon}{C}\right)^m(n^{-s})^m(1+Q)+\underbrace{\left(\frac{\varepsilon}{C}\right)^m(n^{-s})^m R}_{\mathrm{val}>q}.$$

Putting this together, one gets, for any $t\in[-u,u]$,

$$|S_p(f_\alpha,t)-S_q(f_\alpha,t)|=|\alpha_m|(\varepsilon/C)^m|S_K(Q,t)-S_1(Q,t)|>u.$$

This achieves the proof of Theorem 5. ∎

Remark 7: This method is not specific to Dirichlet series. A suitable adaptation gives the following result: given a subset E of $\mathbb{T}$ of measure 0, there exists a function $f\in C(\mathbb{T})$ such that, for any $g\in A(f)\backslash\{0\}$, the Fourier series of g diverges on E. In particular, we replace Lemma 6 by the polynomials which appear in the Kahane–Katznelson proof that every such a set is a set of divergence for some Fourier series (see [16] or [17]). This contains partly the result of [3], where it is proven in addition that we may construct an infinitely generated and dense algebra. However, we have obtained a new information. For quasi-all functions f in $C(\mathbb{T})$, the algebra $A(f)$ consists of functions whose Fourier series diverges on E.

Remark 8: As mentioned in the introduction, Baire's Theorem is often used to exhibit queer functions. This seems to be the first use of it to produce an algebra, or even a subspace, of such functions.

4. Continuous interpolating functions

Benyamini [8] proved that there is a real-valued continuous function f on $\mathbb{R}$ which interpolates every bounded sequence (i.e. for each bounded sequence $\{y_n\}_{n\in\mathbb{N}}$ there is a point $t \in \mathbb{R}$ such that $f(t+n) = y_n$ for all $n \in \mathbb{N}$). His proof uses Alexandroff–Hausdorff theorem that every compact metric space is a continuous image of the Cantor set. Using the terminology introduced by Gurariy in the late eighties (and then used in [2], [10] and [13]), we first prove here that the set of real-valued continuous interpolating functions is lineable: this set contains a linear subspace (except for the zero function) of the largest possible dimension, $2^{\aleph_0}$. The following proof is based on the original idea of Benyamini.

THEOREM 9: *The set of real-valued continuous functions on $\mathbb{R}$ which interpolate every bounded sequence contains a linear subspace (except for the zero function) of dimension $2^{\aleph_0}$.*

Proof: Let Δ be the triadic Cantor set, K be the compact metrizable set $\prod_{n\in\mathbb{N}}[-n,n]$ (endowed with the product topology), $\phi\colon \Delta \to K$ be a continuous and surjective function (given by Alexandroff–Hausdorff theorem) and $\phi_n := p_n \circ \phi\colon \Delta \to [-n,n]$ (where p_n is the continuous projection from K to $[-n,n]$).

For $k \in \mathbb{N}\setminus\{0\}$, let us put $U_k :=]1/(k+1), 1/k[$ and let $\Delta_k \subset U_k$ be a Cantor set, identified with Δ through the homeomorphism $I_k : \Delta_k \to \Delta$. Using Tietze Extension Theorem, we construct continuous functions $\varphi_k\colon \mathbb{R} \to \mathbb{R}$ such that:

(i) $\varphi_k := \phi_n(I_k(x-n))/k$ for $x \in \Delta_k + n$.

(ii) $\varphi_k(x) := 0$ if $x \notin \bigcup_{n\in\mathbb{N}}(U_k + n)$.

(iii) $|\varphi_k(x)| \le n/k$ on the interval $[n-1, n]$.

Using a result of Sierpinski (see [11] for example), let $\{N_\alpha\}$ be a family of $2^{\aleph_0}$ almost disjoint infinite subsets of $\mathbb{N}$ (i.e. $N_\alpha \cap N_\beta$ is finite whenever $\alpha \neq \beta$) and let us put $\varphi_\alpha := \sum_{k\in N_\alpha}\varphi_k$. Since the functions φ_k have disjoint supports, these functions φ_α are well-defined and linearly independent. They are also continuous on $\mathbb{R}$. Let us prove that every (non-zero) finite linear combination $f := \sum_{\alpha\in I} a_\alpha\varphi_\alpha$ interpolates every bounded sequence of real numbers. Let us fix $\alpha \in I$ with $a_\alpha \neq 0$. The almost disjointness of the N_α's implies that there is $k \in N_\alpha \setminus \bigcup_{\alpha\neq\beta\in I} N_\beta$ and then $f = a_\alpha\varphi_k$ on $\bigcup_{n\in\mathbb{N}}(\Delta_k + n)$. Let $y = \{y_i\}_{i\in\mathbb{N}}$ be a bounded sequence. There exists $i_0 \in \mathbb{N}$ such that, for every $i \in \mathbb{N}$, $|k/a_\alpha y_i| \leq i_0$. Let us define the sequence $z = \{z_i\}_{i\in\mathbb{N}}$ by:

$$z_i := \begin{cases} (k/a_\alpha) y_{i-i_0} & \text{if } i \geq i_0, \\ 0 & \text{otherwise.} \end{cases}$$

This sequence z belongs to K. So, there exists $s_0 \in \Delta$ such that $\phi(s_0) = z$. Let us define $t_0 := I_k^{-1}(s_0) + i_0 \in \bigcup_{n\in\mathbb{N}}(\Delta_k + n)$. We have, for all $n \in \mathbb{N}$:

$$f(t_0 + n) = a_\alpha \varphi_k(t_0 + n) = a_\alpha \frac{\phi_{n+i_0}(s_0)}{k} = \frac{a_\alpha}{k} z_{n+i_0} = y_n,$$

which proves that f interpolates the sequence y. ∎

The question of existence of algebras of continuous interpolating functions clearly imposes working with complex-valued functions. As in [4], we construct “big” algebras using rings of polynomials in complex variables. Let us first observe that:

Lemma 10: *If $P \in \mathbb{C}[X]$ is a non-constant polynomial and f is a complex-valued continuous function on $\mathbb{R}$ which interpolates every bounded sequence of complex numbers, then $P(f)$ also satisfies this property.*

Proof: Let $f\colon \mathbb{R} \to \mathbb{C}$ be a complex-valued continuous interpolating function, $P \in \mathbb{C}[X]$ be a non-constant polynomial (of degree $n \geq 1$), $y = \{y_k\}_{k\in\mathbb{N}}$ be a bounded sequence of complex numbers, and let us prove that $P(f)$ interpolates this sequence. For all $k \in \mathbb{N}$, let us consider $z_k \in \mathbb{C}$ a solution of $P(x) = y_k$. The sequence $\{z_k\}_{k\in\mathbb{N}}$ is bounded. Since f is interpolating, there exists $t \in \mathbb{R}$ such that, for all $k \in \mathbb{N}$, $f(t + k) = z_k$. Then, for all $k \in \mathbb{N}$, $P(f)(t + k) = y_k$, which proves that $P(f)$ is interpolating. ∎

This implies that if f is a complex-valued continuous interpolating function, then the non-trivial algebra $A(\{f\})$ is contained (except for the zero function) in the set of all complex-valued continuous interpolating functions. This algebra is infinite dimensional as a vector space (since $\{f^n\colon n \in \mathbb{N} \setminus \{0\}\}$ is a countable family of linearly independent functions) but it is generated (as an algebra) by the single function f. In order to prove that this set contains an algebra with infinitely many algebraically independent generators, since all the subalgebras of $\mathbb{C}[X]$ are finitely generated, we have to work with polynomials in several variables. We will use the fact that the ring $\mathbb{C}[X, Y]$ of polynomials in two complex variables contains a subalgebra $\mathcal{A}$ whose minimal system of generators is $S = \{xy^n\colon n \in \mathbb{N}\}$.

THEOREM 11: *The set of complex-valued continuous functions on $\mathbb{R}$ which interpolate every bounded sequence contains an algebra with infinitely many algebraically independent generators.*

Proof: Let Δ be the triadic Cantor set, $K := \prod_{n\in\mathbb{N}} \overline{B_{\mathbb{C}}(0;n)}$, $\phi\colon \Delta \to K$ be a continuous and surjective function and $\phi_n := p_n \circ \phi\colon \Delta \to \overline{B_{\mathbb{C}}(0;n)}$.

Let us define the set $N := \{k \in \mathbb{N} : k = 2^{k_1}3^{k_2}; k_1, k_2 \in \mathbb{N}\}$. With φ_k as in the proof of Theorem 9 (but now with the complex version of ϕ), let us take:

$$f_1 := \sum_{k\in N} (\varphi_k)^{k_1},$$
$$f_2 := \sum_{k\in N} (\varphi_k)^{k_2}.$$

Let $P \in \mathbb{C}[x, y]$ be a non-constant polynomial. As pointed out in [4], there exists m_1, $m_2 \in \mathbb{N}$ such that the polynomial $Q(z) := P(z^{m_1}, z^{m_2})$ is non-constant. Let $m := 2^{m_1}3^{m_2}$. Then,

$$P(f_1, f_2) = P(\varphi_m^{m_1}, \varphi_m^{m_2}) = Q(\varphi_m) \quad \text{on } \bigcup_{n\in\mathbb{N}} (\Delta_m + n).$$

Since φ_m interpolates every bounded sequence of complex numbers, by Lemma 10, $P(f_1, f_2)$ is also interpolating. To conclude the proof, we consider the subalgebra $\{P(f_1, f_2)\colon P \in \mathcal{A}\}$, which is infinite dimensional as a vector space and infinitely generated as an algebra. ∎

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